Reinterpretation of a Kardar-Parisi-Zhang equation-based classification

N. Neshkov
Department of Physics, Brown University, Providence, Rhode Island 02912
(Received 1 June 1999; revised manuscript received 30 November 1999)

The velocity versus tilt behavior, on which the Kardar-Parisi-Zhang (KPZ) equation-based quenched Edward-Wilkinson–directed percolation depinning (QEW-DPD) classification scheme of models of rough interface growth in a medium with quenched disorder is based, is reinterpreted. The consideration of the screen grid of pixels in computer simulation interface propagation allows an explanation of tilt-velocity behavior without assuming vanishing or divergence in the depinning transition of the KPZ parameter λ which is found to have a unique, measurable value in DPD. Random field Ising model-like velocity-tilt behavior in the QEW class is shown to either correspond to zero λ. The possibility of obtaining low velocity $v(m)$ behavior from the KPZ method raises some interesting questions about the depinning transition.

PACS number(s): 47.55.Mh, 68.35.Ja

The quenched Edward-Wilkinson–directed percolation depinning (QEW-DPD) scheme was recently introduced for the classification of models of interface motion in a medium with quenched disorder. The scheme is based on the quenched Kardar-Parisi-Zhang (KPZ) equation [1]

$$\partial h/\partial t = F + \eta(x,h) + \nabla^2 h + \lambda(\nabla h)^2. \tag{1}$$

''Quenched'' means that the noise η has only spatial dependence, no time dependence, while λ is related to lateral local growth. Setting $\lambda=0$ gives the QEW equation.

Based on the behavior of average interface velocity v versus global tilt m, Amaral *et al.* [2] classified models as either having negligible or significant nonlinear $\lambda(\nabla h)^2$ term. The former models belong to a QEW universality class, named after the equation, and the latter belong to the class DPD, bearing the name of a representative model [3].

The classification is explained in the KPZ context by representing the tilt as the transformation :

$$h(x,t) \rightarrow y_{gl}(x,t) \equiv h(x,t) + mx, \tag{2}$$

which, when applied to the spatially averaged KPZ equation, is claimed to lead to parabolic tilt-velocity behavior:

$$v(m) = \overline{\frac{\partial y_{gl}}{\partial t}} = \overline{F + \eta + \nabla^2 h + \lambda(\nabla h)^2} = v_o + \lambda m^2. \tag{3}$$

Such parabolic behavior is never observed.

The $v(m)$ behavior in the QEW class is either horizontal lines (i.e., v independent of m) or as in Fig. 2, for the random field Ising model (RFIM) [4]. For the DPD model $v(m)$ is as in Fig. 1.

From the suggestion of parabolic dependence, the $v(m)$ behavior in the figures has been interpreted as either $\lambda \rightarrow 0$ or $\lambda \rightarrow \infty$ as $v \rightarrow 0$ (the depinning transition). The vanishing of λ is in accordance with a previous interpretation from Narayan and Fisher [5]. Its divergence was posed as an unsolved problem by Amaral *et al.* in the paper introducing the classification.

In the present work a reinterpretation of $v(m)$ behavior, resolving the problem, is suggested. In addition, this reinterpretation allows a measurement of the λ parameter in DPD. Arguments are given suggesting that $\lambda=0$ identically in the QEW universality class.

The reinterpretation is based on the ambiguity about the frame in which the $\partial h/\partial t$ term in the KPZ scheme is measured. In computer simulations there are two different natural directions for the orientation of this frame's vertical. These are the direction determined by the vertical of the grid of pixels on the screen h, and the direction, normal to the interface orientation y. These correspondingly determine *the grid frame* and *the interface frame*. The interface frame can be determined either locally, at each interface point, by the orientation of the local interface slope, or globally, by the orientation of the interface average.

The interplay between the *local* interface and the grid frames is implicit in the original KPZ derivation. On the other hand, to our knowledge the interplay between the grid and the *global* interface frame has never been considered before, because the two frames coincide unless tilt is imposed through helicoidal boundary conditions. Measurements of velocity in simulations are always done in the grid frame.

When local growth is also along the grid frame vertical h and stays along h irrespective of global or local tilt as in the model of random deposition [6], velocity is independent of tilt and $\lambda=0$ because there is no lateral growth. Such models are QEW models.

The assumption that growth takes place in the local interface frame (i.e., normal to the interface) leads to the KPZ equation because then [1]:

$$\frac{\delta h}{\delta t} = \frac{\delta y_{loc}}{\delta t} \frac{1}{\cos\varphi_{loc}}, \tag{4}$$

where $\cos\varphi_{loc}$ is the angle between the grid and the local interface frames. Note that $\delta h/\delta t$ is not tilt independent in this case.

The nonlinear KPZ term is produced from the fact that, locally,

$$v_0/\cos\varphi_{loc} = v_0\sqrt{1+(\nabla h)^2} \approx v_0 + \frac{v_0}{2}(\nabla h)^2, \tag{5}$$

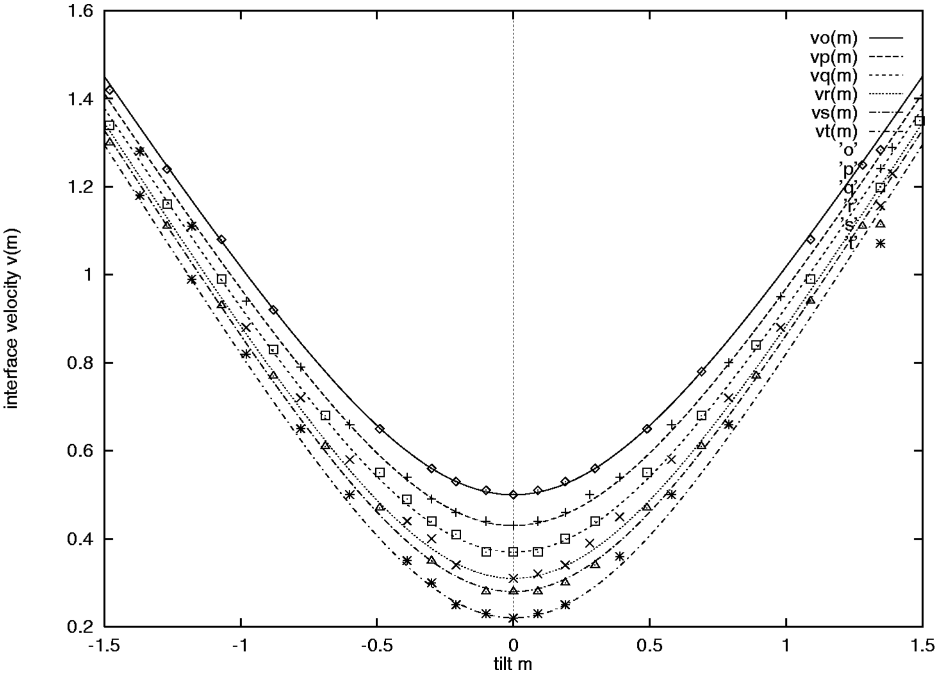

FIG. 1. Fit of $(v_0+\lambda m^2)/(\sqrt{1+m^2})$ against the data for a model in the DPD universality class with $\lambda=0.94$. Fitting with the Levenberg-Marquardt algorithm with free parameters v_0 and λ produced the values of v_0 read as $v(m=0)$ and $\lambda=0.94$, both with a precision ± 0.01.

where $\approx$ is in assumption of small local tilt ∇h. The smallness is assured by the additive relaxation term $\nabla^2 h$.

When global tilt is imposed, the assumption of smallness is violated in the grid frame, because all local tilts there become

$$\nabla h \rightarrow \nabla(h+m\mathbf{x}) \equiv \nabla y_{gl}, \tag{6}$$

with y_{gl} the same as in Eq. (2) and m not small.

The assumption of small tilt and, hence, the KPZ equation is then only valid in the global interface frame so that, to obtain the measured in the grid frame behavior, a transformation back by a factor $\cos\varphi_{gl}=1/\sqrt{1+m^2}$ has to be performed before taking the average, giving

$$v(m)=\overline{\frac{\partial h}{\partial t}}=\overline{\frac{\partial y_{loc}}{\partial t}\cos\varphi_{gl}}=\frac{v_0+\lambda m^2}{\sqrt{1+m^2}}. \tag{7}$$

A fit of this expression against the data for the DPD model with $\lambda=0.94$ is shown in Fig. 1. Fitting with the Levenberg-Marquardt, (LM) algorithm with free parameters v_0 and λ produced both the values of v_0, read as $v(m=0)$ and $\lambda=0.94$ with precision ± 0.01. The result from LM fitting of the parabolic behavior (3) is unsatisfactory (Fig. 3). The imprecision in the values of v_0 obtained by the fit was greater by an order of magnitude for this case.

In the context of this interpretation λ does not diverge, but has a set, measurable value. To the extent that by virtue

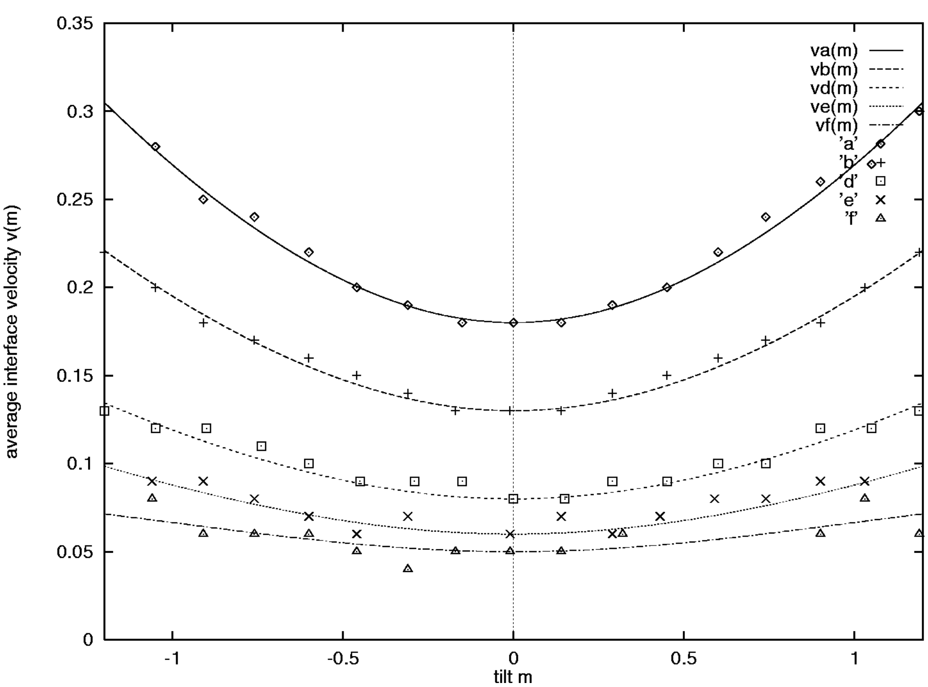

FIG. 2. Fit of $v(m)=v_0\sqrt{1+m^2}$ against $v(m)$ behavior for the random field Ising model, which is in the QEW-2 universality class. For each graph the value of v_0 was read off from the data as $v(m=0)$.

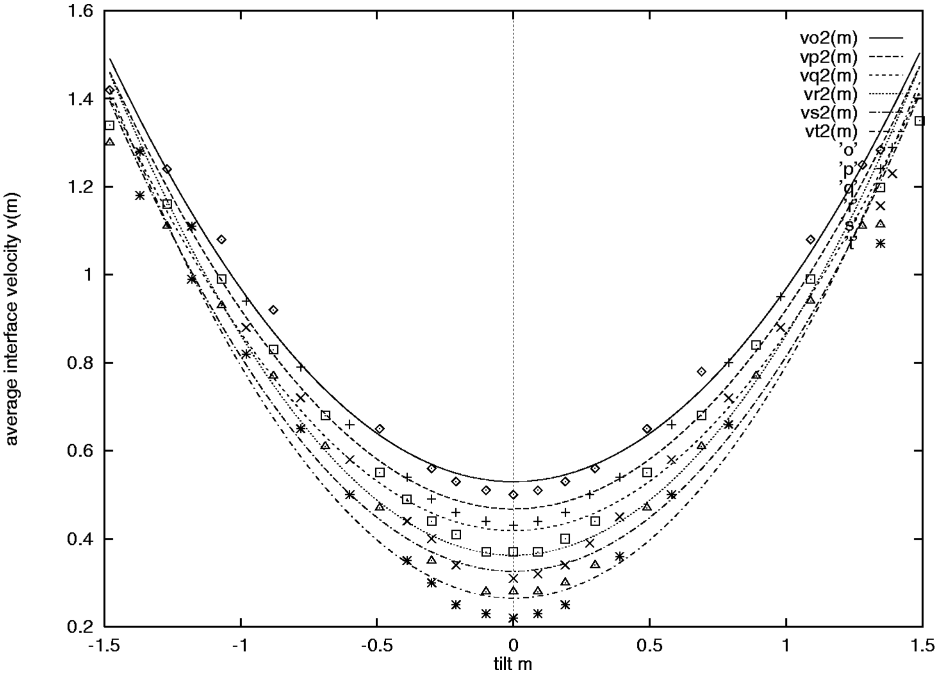

FIG. 3. LM fitting of the parabolic behavior (3) produces an unsatisfactory result. The imprecision in the values of v_0 obtained by the fit was greater by an order of magnitude for this case.

of Eq. (3) the interpretation is derived from the KPZ model, it confirms the applicability of the equation to DPD models.

The argument leading to the interpretation will be even more convincing if it can also explain the RFIM-like $v(m)$ dependence as shown in Fig. 2 without assuming λ to be vanishing.

This can be achieved by considering the equivalence of the above argument for the DPD class to the natural assumption that upon the imposition of global tilt the local direction of growth at each point is rotated by the angle $\varphi_{gl} = \arccos(1/\sqrt{1+m^2})$. The situation will be referred to as *local isotropy*.

An alternative, equally natural situation is that of *global isotropy*, when it is the velocity of the interface average which rotates by the angle of φ_{gl} upon imposition of global tilt. In this case the helicoidal boundary conditions will act as an elevator for the portion of the interface which tends to move off the screen and the measured in the grid frame velocity for tilted interfaces will be *greater* by a factor of $1/\cos\varphi_{gl}$:

$$v(m) = v_0\sqrt{1+m^2}. \tag{8}$$

A fit of this expression against the data for actual v(m) behavior is shown in Fig. 1; for each graph the value of v_0 was read off from the data as $v(m=0)$. The read value of v_0 is the *only* parameter in the fit. Levenberg-Marquardt fitting of the expression $v(m) = (v_0 + \lambda m^2)\sqrt{1+m^2}$ produced the read values of v_0 and $\lambda = 0$ with precision ± 0.001.

Such $v(m)$ dependence can be explained by assuming that local growth is always oriented along the vertical of the *global* interface frame irrespective of local tilt. Then in the global interface frame there is no lateral growth and $\lambda = 0$, so that the interface motion satisfies the QEW equation in the global interface instead of the grid reference frame.

This assumption justifies the classificaion of the RFIM in the QEW class, and explains the obtained value of $\lambda = 0$. The experimental results of [7] are in complete agreement with Eq. (8) and $\lambda = 0$.

It is good to note at this point that in the DPD universality class the combination of an assumption of parabolic behavior as in Eq. (2), with the assumption of a divergence of λ,

$$\lambda \sim f^{-\phi}, \tag{9}$$

in the depinning transition leads to the equation

$$v(m,f) \propto f^{\theta} + a f^{-\phi} m^2, \tag{10}$$

predicting a crossing of the graphs $v(m,f_1)$ and $v(m,f_2)$ corresponding to two different driving forces $f_1 > f_2$ [2]. Such a ''crossing effect'' is not observed for the data at hand and does not occur if $v(m)$ is assumed to be in the form of Eq. (7) with constant λ.

From Eq. (7), for large tilts $v(m)$ is linear in m,

$$v(m) \sim \lambda m, \tag{11}$$

which agrees with the observed DPD behavior in Fig. 1. This asymptotic dependence precludes the ''crossing effect.''

In accordance with Eq. (7) and the above argument the exponent ϕ, introduced to account for the crossing effect, should be $\phi = 0$. More interesting but more difficult to interpret is the implication of Eq. (7) for the interpretation of the θ exponent.

The exponent is obtained from the assumption that the dependence of velocity v on driving force F falls into one of two regimes: $v \sim F$ (moving phase) or $v \sim f^{\theta}$ (*crossover region*), where $f = (F - F_c)/F_c$, F_c—a critical force at which the interface stops propagating (i.e. it is *pinned*, hence *depinning transition*). The moving phase can be interpreted by assuming temporal noise dependence, $\eta \equiv \eta(\mathbf{x},t)$, which KPZ described adequately. It is not yet understood whether the depinning transition can be captured by the KPZ approach with quenched, i.e. spatial noise dependence: $\eta \equiv \eta(\mathbf{x},h)$.

In the present context the possibility of obtaining expression (7) from the KPZ model for the lowest velocities v_0 in Fig. 1 suggests the validity of the KPZ approach in this region, from which we have the following possibilities.

(1) The depinning transition affects only the magnitude of the local growth velocity, but not the relation between lateral and forward growth from which the KPZ equation stems, so that the equation is still effective in the depinning transition region.

(2) Even the lowest velocity in Fig. 1 is still in the moving phase where the KPZ equation is valid.

(3) There is no crossover region in the depinning transition, and all depinned interfaces obey the KPZ equation. This is equivalent to assuming $\theta = 1$, a result obtained previously for globally tilted interfaces by Tang *et. al.* [8], who, surprisingly enough, derived from very different considerations of flux lines in a superconductor, a differential equation having a term very similar to Eq. (7).

Of the above claims, (1) can be probed only theoretically; (2) can be checked numerically by going to ever lower velocities (there being no telling how low though); and (3) is subject to experimental verification by checking the $v \sim f^{\theta}$ dependence in the low velocity region, (this can be done by digitizing and averaging a CCD camera image of imbibition in a porous medium driven by adjustable pumping rate F).

The chief disadvantage of this classification scheme, carried over to its reinterpretation is that it carries no direct information about the roughness exponents, and thus cannot explain the exponent $\alpha = 0.63$ observed for models in DPD [2]. This exponent disagrees with theoretical [1] predictions giving $\alpha = 0.5$, agrees with some older numerical [9] and experimental [3] works, and disagrees with newer, very careful [10] experimental results for KPZ-related situations. In this context studies of the roughness exponent for globally tilted interfaces can be very revealing for the classification.

Also, to our knowledge, until the present work the $v(m)$ behavior leading to the classification was derived from the KPZ equation but interpreted as taking place in a crossover region not adequately described by KPZ—a contradiction which our reinterpretation resolves together with the problem of the λ-parameter divergence.

The author would like to thank Professor J.M. Kosterlitz for truly invaluable discussions and help and Professor Andrew Dougherty from the Lafayette College Physics Department for an opportunity to learn about this area of Physics. Routine but useful help from M. Anatchnova is acknowledged.

[1] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).

[2] L. A. N. Amaral, A.-L. Barabasi, H. A. Makse, and H. E. Stanley, Phys. Rev. E **52**, 4087 (1995).

[3] S. V. Buldyrev, A.-L. Barabasi, F. Caserta, S. Havlin, H. E. Stanley, and T. Vicsec, Phys. Rev. A **45**, R8313 (1992).

[4] H. Ji and Mark O. Robbins, Phys. Rev. A **44**, 2538 (1991); E. Ising, Ann. Phys. (Leipzig) **31**, 253 (1925).

[5] O. Narayan and D. S. Fisher, Phys. Rev. B **48**, 7030 (1993).

[6] F. Family, J. Phys. A **19**, L441 (1986).

[7] R. Albert, A.-L. Barabasi, N. Carle, and A. Dougherty, Phys. Rev. Lett. **81**, 2926 (1998).

[8] L.-H. Tang, M. Kardar, and D. Dhar, Phys. Rev. Lett. **74**, 920 (1995).

[9] Z. Csahok, K. Honda, E. Somfai, M. Vicsek, and T. Vicsek, Physica A **200**, 136 (1993).

[10] J. Maunuksela, M. Myllys, O. P. Kähkönen, J. Timonen, N. Provatas, M. J. Alava, and T. Ala-Nissila, Phys. Rev. Lett. **79**, 1515 (1997).